**RSTDPT Interaction Picture**

**The (time-development) S-matrix**

So our general equation governing the time-development of a particle is:



But this isn’t directly useful. Let’s consider our Hamiltonian again. Usually it consists of a part that is easy to solve exactly, and a part that isnt.



where H0(t) is the part of the Hamiltonian that we could solve if it were alone (usually time-independent), and V is an extra term, which may be time-dependent, that prevents solution. We could put this equation in the position basis and translate it into a partial differential equation. Then we could attempt a perturbative solution of the resulting partial-differential equation. But it is cleaner if we leave it as an operator equation for now, and then, once we have our result in hand, translate it to position space if we like. So in that case, what we would like, is to determine what the time-development operator, U(t), looks like in this case with the extra V(t) term in the Hamiltonian. For once found, we can say that:



The way we will find U(t) is by plugging it into the Schrodinger equation and determining an equation for it, like we did in a previous lecture. So doing…



again, this equation is true regardless of the initial state, |ψ0>. So it is true that:



So what is the solution to this equation? We should solve it perturbatively as was done before, treating V(t) as a perturbation this time. So stick a λ on it, and expand our time-evolution operator in a power series in λ.



(I’m putting the former superscript (n) in the subscript so it doesn’t get in the way of other things). Plugging it in we get:



Equating power by power:



The first equation it is assumed we know how to solve, and the answer is (see the file in Foundations folder):



which will usually be, since H0(t) is usually time-independent,



The second equation is a linear inhomogeneous 1st order equation.



To solve it we need to multiply by the integrating factor etc. Note that the integrating factor I(t) = Texp(-∫H0/iћ dt) is actually just U0†(t). So we have:



Finally noting that the inverse of U0† is just U0 itself we have the first order term,



Alright. What’s the second term? It follows the equation,



We don’t have to do any more work since the second equation is of the same form as the first order equation. If we follow the previous analysis, then we can see that:



Similarly, the third order term obeys the equation,



and the solution, extending the analogy, will be:



etc. So we can see that going out to nth order, the expression will be:



The terms are getting a little out of hand, but we see a repeating expression. So let us define:



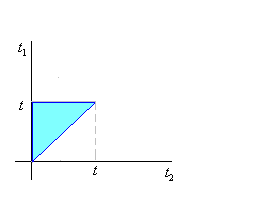
This is called the ‘interaction picture’ V(t). Then the perturbative expansion for the time-evolution operator can be written:



The term in brackets is called: S(t). So this is our answer. Since this is such an important result, lots of work has been done to put S(t) in the ‘nicest’ form possible. So I will show you the form that this is usually put in – just so you’re aware of it. But for practical purposes, the expression above is the one we will use. That said, take a look at the second term in the brackets.



The integration region runs over the following triangle in (t1,t2) space



But the integrand is symmetric since if we switch t1 and t2 we get the same operators back – almost. Actually switching t1 and t2 doesn’t quite give us the same thing back because the operators V(t1) and V(t2) usually don’t commute.



But the integrand will be symmetric if we just agree to keep the ordering of the operators the same. Note how t1 > t2, and how the order of the operators is VI(t1)VI(t2) so that the ‘later’ time occurs on the left. If we agree to always keep the operators in that order ‘later times to the left’, then the integrand will be symmetric about the diagonal and we can extend the region of integration to the lower triangle (and divide by 2 to compensate). So then we can say:



where T is the ‘time ordering operator’ which orders the operators according to the latest time arguments to the left. For example,



Or formally,



Similarly, we can write the third order term as:



and generally,



and with this we may write the bracket as:



where the last expression is to be interpreted as signifying the previous line. So altogether we can say that our time-evolution operator looks like (setting λ = 1)



For what it’s worth, should probably mention that ODE obeyed by S operator is clearly,



**Special case**

Suppose that the perturbation VI(t) is an operator that commutes with itself at different times. Then what is S(t)? Well we’ll have:



where we have used the fact that even after the time-ordering operator has arranged the operators, VI(t), in the proper order in line 2, we can rearrange them without error because these operators commute with each other and it doesn’t matter what order we multiply them in. A special special case of this would be when VI(t) is just a scalar function of time.

**Generalization to non-zero starting times**

It’s trivial to go back and change our initial time to t0, rather than 0. In that case we’re trying to solve,



and the solution would be: |ψ(t)> = U(t,t0)|ψ0>, where,



and,



**Mixed Phase Convention**

It seems, especially when we have perturbations that start at t0 = -∞, we use a different phase convention for the time-evolution operator, to avoid pesky infinities that don’t really matter usually because they’re just phases. So, above, our phase is implicitly zero at t = t0. Instead, we’d like our phase to be zero at t = 0, in the absence of the perturbation – or in other words, in the absence of a perturbation, the time-development should look like it would normally starting at time t = 0. And so we’ll want to solve the problem:



which amounts to |ψ(t)> = U´(t,t0)|ψ0>, where U´(t,t0) satisfies the equation:



Then we’d make the perturbative expansion: U´ = U´0 + λU´1 + λ2U´2 + … resulting in equations:



And just like before, we’ll get:



We’ll note that the initial conditions are satisfied by the first term alone – the other guys go to zero when t → t0. So skipping to the end, we have:



and,



and I guess we can straight-forwardly accommodate time-dependent H0 here, in the expected way.



How does this U´(t,t0) compare with the usual, especially when taking expectations, like <ψ(t)|A|ψ(t)>? Well, we should also be able to say, given our problem setup,



that, the solution is:



since it solves the ODE, and the initial condition. This would imply that:



And so if we wanted to find the expectation:



we can see that it would be the same as:



If the states against which we take the expectation are H0 eigenstates. This is predominantly the case. What about projections? This would be:



If |ψ0> is an eigenstate of H0, then this will just introduce an extra phase. And this will even go away if we take the modulus.

**Schrodinger equation for interaction picture wavefunction**

The Schrodinger equation for the wavefunction simplifies if we factor out the time-development we know how to solve. Consider the following definition:



and let’s see what the Schrodinger equation is for |ψ(t)>I. So let’s fill this expression into Schrodinger equation:



So we see the interaction picture wave-function satisifes a simpler PDE.



We implicitly used these manipulations when we solved the spin-in-magnetic field problem.

**Heisenberg equation for interaction picture operator**

Let’s do a similar thing for operators. And consider the following definition:



Taking the derivative of AI(t), we see that its ODE is simply,



So it appears that in the interaction picture, the time-dependence is split up and the operator gets the ‘easy’ part, while the wavefunction gets the ‘hard’ part. In this sense, the interaction picture is kind of ‘in between’ the Schrodinger picture (where wavefunction gets all the time development), and the Heisenberg picture (where the operator gets all the time-development).